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Analysis of perfectly bonded wedges and bonded wedges with an interfacial crack under antiplane shear loading

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Abstract

Antiplane shear deformation of perfectly bonded wedges as well as bonded wedges with an interface crack are studied in this paper. The solution of governing differential equations is accomplished by means of the Mellin transform. For two edge-bonded isotropic wedges with perfect bonding along the common edge, closed form solutions are obtained for stress fields and analytical relations are given for the strength of singularity at the apex. However, for bonded wedges with an interfacial crack, first it is necessary to express the traction-free condition of the crack faces in the form of a singular integral equation which is done in this paper by describing an exact analytical method. The resultant singular integral equations are then solved analytically and the obtained results including the stress intensity factors at the crack tips are plotted. A comparison of the results in the special cases shows a complete agreement with those cited in the literature. However, when the crack tip coincides with the wedge apex, a strength of singularity of unity has been observed. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

The stress analysis in a wedge with infinite radius has been considered by various investigators. Tranter (1948), by employing the Airy stress function and using the Mellin transform, solved the plane elasticity problem of an infinite isotropic wedge. Then, Williams (1952) studied the stress singularities at the wedge apex by using the eigenfunction expansion method. Later on, in a series of papers, Bogy (1972) and Kuo and Bogy (1974a, 1974b) employed a complex function representation of the solution in conjunction with a generalized Mellin transform to analyze stress singularities in an anisotropic wedge. The stress distribution in a wedge with finite radius subjected to antiplane shear deformation was obtained by Kargarnovin et al. (1997). They also extracted the strength of stress singularities at the wedge apex, under different boundary conditions, as a function of apex angle. Afterwards, Shahani (in

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an accepted for publication paper), by defining some complex integral transformations, solved the antiplane deformation problem of anisotropic finite wedges and derived analytical relations for the strength of stress singularities in terms of 'transformed apex angles'.

The problem of finding the stress singularities at the apex of a bi-material wedge was examined by Bogy (1971) and Dempsey and Sinclair (1979, 1981), for the in-plane problems. Ma and Hour (1989) studied the asymptotic behaviour of the stress components in the vicinity of the apex of a bi-material wedge. They restricted themselves to the derivation of the equation of the poles in the Mellin transform domain and analytical relations for the orders of stress singularities in special cases.

The stress singularities due to the existence of an edge crack in simple and bi-material wedges were considered by Kargarnovin et al. (1997), Ting (1986) and Ma and Hour (1989). They found that the analysis of the problem of an edge crack was the special case of their analysis of a simple isotropic or a bi-material wedge.

Unlike the analytical and practical interest, the problem of bonded wedges with an interface crack was not the subject of the investigations made on bonded materials, because of analytical difficulties. An analytical approach to this problem, under antiplane shear loading, was done by Erdogan and Gupta (1975). The major task of this paper is to express the prescribed boundary condition on the crack region in the form of a singular integral equation. For this purpose, dual integral equations were extracted from the relations which were resulted from the solution of the equilibrium equations and their related boundary conditions, by utilizing the Mellin transform. Then, using several series expansions on different terms of the integrand and adding and subtracting the leading terms of these series to and from the integrand, the attempt was made to extract an equation in the form of a standard singular integral equation. This singular integral equation was then solved with an approximate numerical method.

The analysis of perfectly bonded wedges as well as bonded wedges with an interface crack, under antiplane shear loading is the subject of the present investigation. The antiplane shear tractions act on the edges of the bi-material wedge and a traction-free condition is imposed on the crack faces. The tractions are assumed to act concentrically which allows the solutions to be used as the Green's function for the analysis of a wedge under general distribution of traction. The solution is accomplished by employing the Mellin transform. The problem of perfectly bonded wedges is analyzed as a special case of the crack problem by setting the coordinates of the crack tips equal. The full field solution is obtained for stress components and analytical relations are given for the strength of singularity at the apex. In the presence of an interfacial crack, we will describe an exact analytical technique for extracting the singular integral equation from the boundary condition prescribed on the crack faces in the case of wedges with equal apex angles. This singular integral equation will then be solved analytically and the results together with the stress intensity factors at the crack tips will be plotted. Comparing results of special cases, with those published in the literature shows a complete agreement. However, when the crack tip coincides with the wedge apex, a singularity of the order unity has been detected.

2. Formulation and problem solution

A bi-material wedge composed of two bonded isotropic wedges with apex angles θ_1 and θ_2, shear moduli μ_1 and μ_2 and infinite length in the direction perpendicular to the plane of the wedge is considered as shown in Fig. 1. Because of imperfect bonding, a crack exists along the common edge. Choosing the common edge as the reference axis for defining the coordinate θ, the crack lies on the line $\theta = 0$ between the radii $r = a$ and $r = b$. The condition of antiplane shear deformation is imposed on the composite wedge and traction-traction boundary conditions occur on the edges of the composite wedge; however, on the faces of the crack the traction-free condition is applied. In such conditions, the only non-zero displacement component is the out-of-plane component, W, which is a function of in-plane

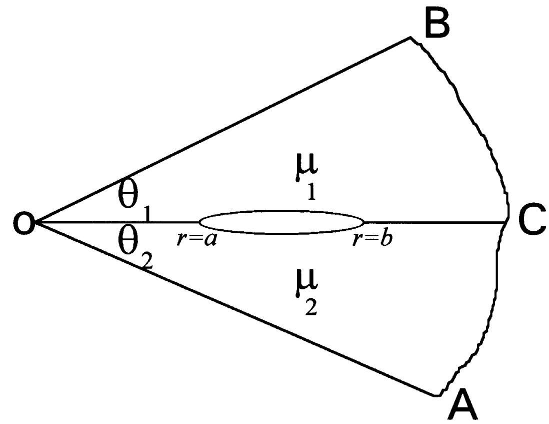

Fig. 1. Schematic view of a bi-material wedge with an interface crack.

coordinates r and θ. Therefore, the nonvanishing stress components are $\tau_{rz}(r,\theta)$ and $\tau_{\theta z}(r,\theta)$. The constitutive equations for isotropic materials undergoing antiplane deformation reduce to

$$\tau_{rz}^{k}=\mu_k\frac{\partial W_k}{\partial r}$$

$$\tau_{\theta z}^{k}=\frac{\mu_k}{r}\frac{\partial W_k}{\partial\theta},\quad k=1,2 \tag{1}$$

for each of the two wedges denoted by superscripts or subscripts $k = 1, 2$. In the absence of body forces, by making use of (1), the equilibrium equation in terms of displacement appears as

$$\nabla^2 W_k=0,\quad k=1,2 \tag{2}$$

The differential equations (2) must be solved under the following boundary conditions:

$$\tau_{\theta z}^{1}(r,\theta_1)=P\delta(r-h) \tag{3}$$

$$\tau_{\theta z}^{2}(r,-\theta_2)=P\delta(r-h) \tag{4}$$

$$W_1(r,0)=W_2(r,0),\quad 0\leqslant r\leqslant a,\quad b\leqslant r<\infty \tag{5}$$

$$\tau_{\theta z}^{1}(r,0)=\tau_{\theta z}^{2}(r,0) \tag{6}$$

$$\tau_{\theta z}^{1}(r,0)=\tau_{\theta z}^{2}(r,0)=0,\quad a\leqslant r\leqslant b \tag{7}$$

In relations (3) and (4), δ denotes the Dirac-Delta function. It is worth mentioning that the choice of these two boundary conditions, leads to the Green's function solution for the problem. Also, in eqns (3) and (4), h is the location of the application of the concentrated tractions which may vary from zero to infinity. Without loss of generality of the problem, here we suppose that $h\geqslant b\geqslant a$.

The solution to this problem is well accomplished by means of the infinite Mellin transformation, which is defined as

$$M\big[W(r,\theta),S\big]=W^*(S,\theta)=\int_0^{\infty}r^{S-1}W(r,\theta)\,\mathrm{d}r \tag{8}$$

where S is a complex transform parameter. The inversion of this transformation is represented by

$$M^{-1}\big[W^*(S,\theta),r\big]=W(r,\theta)=\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}r^{-S}W^*(S,\theta)\,\mathrm{d}S \tag{9}$$

The application of the Mellin transform in conjunction with integration by parts on (2) yields

$$\frac{\mathrm{d}^2W_k^*}{\mathrm{d}\theta^2}+S^2W_k^*(S,\theta)=0,\quad k=1,2 \tag{10}$$

provided that

$$\lim_{r\to 0}\left[r^{S+1}\frac{\partial W_k}{\partial r}+r^SW_k(r,\theta)\right]=0,\quad k=1,2 \tag{11}$$

The condition (11) specifies the strip of regularity which is the range of proper values for the real quantity C in the inversion formula (9). The solution to eqn (10) is readily known to be

$$W_k^*(S,\theta)=A_k(S)\sin(S\theta)+B_k(S)\cos(S\theta),\quad k=1,2 \tag{12}$$

Taking the Mellin transform from the boundary conditions which are prescribed on the whole boundary, i.e., (3), (4) and (6) and applying the resultant relations with the aid of the second of eqns (1) on eqn (12) lead to the relations between the coefficients A_k and B_k as follows:

$$B_1(S)=A_1(S)\cot(S\theta_1)-\frac{Ph^S}{\mu_1S\sin(S\theta_1)}$$

$$B_2(S)=-A_2(S)\cot(S\theta_2)+\frac{Ph^S}{\mu_2S\sin(S\theta_2)}$$

$$A_2(S)=\frac{\mu_1}{\mu_2}A_1(S) \tag{13}$$

Making it possible to apply the boundary conditions (5) and (7), the following unknown function may be defined:

$$f(r)=\frac{\partial}{\partial r}\big[W_1(r,0)-W_2(r,0)\big] \tag{14}$$

With this definition, the condition of continuity of displacements outside the crack, eqn (5), becomes

$$f(r)=0,\quad 0\leqslant r\leqslant a,\quad b\leqslant r<\infty \tag{15}$$

Also, the single-valuedness condition of displacements requires that

$$\int_a^b f(r)\,\mathrm{d}r = 0 \tag{16}$$

Applying now the inversion formula (9) on (12), the solution may be obtained as

$$W_k(r,\theta) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} r^{-S}\big(A_k \sin(S\theta) + B_k \cos(S\theta)\big)\,\mathrm{d}S, \quad k = 1, 2 \tag{17}$$

Substituting (17) into (14), taking the Mellin transform from both sides of the resultant equation and using the boundary condition (15), in conjunction with the use of eqn (13), lead to the relation for the determination of A_1 in terms of $f(r)$ as follows:

$$A_1(S) = \frac{\mu_2}{S\big[\mu_2 \cot(S\theta_1) + \mu_1 \cot(S\theta_2)\big]}\left[-\int_a^b f(v)v^S\,\mathrm{d}v + Ph^S\left(\frac{1}{\mu_1 \sin(S\theta_1)} + \frac{1}{\mu_2 \sin(S\theta_2)}\right)\right] \tag{18}$$

Applying the second of eqns (1) with $k = 1$ on (17), together with the use of the first of eqns (13) and eqn (18), we may obtain

$$r\tau^1_{\theta z}(r,\theta) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{P\left(\frac{h}{r}\right)^S \big\{\sin\big[S(\theta_1-\theta)\big] + \sin\big[S(\theta+\theta_2)\big] + (R-1)\sin(S\theta_2)\cos(S\theta)\big\}}{R\cos(S\theta_1)\sin(S\theta_2) + \sin(S\theta_1)\cos(S\theta_2)}\,\mathrm{d}S - \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\int_a^b \frac{\mu^2\left(\frac{v}{r}\right)^S f(v)\sin\big[S(\theta_1-\theta)\big]\sin(S\theta_2)}{R\cos(S\theta_1)\sin(S\theta_2) + \sin(S\theta_1)\cos(S\theta_2)}\,\mathrm{d}v\,\mathrm{d}S \tag{19}$$

in which R is defined as

$$R = \frac{\mu_2}{\mu_1} \tag{20}$$

The first integral in (19) is the solution of a bi-material wedge composed of two isotropic wedges bonded together along a common edge without any crack on the interfacial region ($a = b$ which implies that no crack exists and sets the second integral in (19) equal to zero). The equation for the poles is obtained by putting the denominator of the integrand in (19) equal to zero:

$$R\cos(S_n\theta_1)\sin(S_n\theta_2) + \sin(S_n\theta_1)\cos(S_n\theta_2) = 0 \tag{21}$$

or in a simpler form:

$$\tan(S_n\theta_1)\cot(S_n\theta_2) = -R \tag{22}$$

Eqn (22) has not been solved analytically in general, but it appears to have symmetric real roots $\pm S_n$ with respect to imaginary axis of the complex plane. Moreover, it should be possible to show that the roots are simple poles of the integrands of (19). On the other hand, from the condition (11) and the requirement that the expression for strain energy ought to be integrable in the vicinity of the wedge apex, the strip of regularity becomes $|C| < S_1$, where S_1 is the least of the poles S_n.

3. Analysis of a bi-material wedge with perfect bonding along the interface

As mentioned above, letting $a = b$ in (19) we may obtain the solution of the perfect bonding problem of a bi-material wedge. To find the stress field, contour integration should be used. The integrand is a meromorphic function in S and two different regions should be considered which are: $r \leqslant h$ and $r \geqslant h$. The choice of contour is subjected to the requirement that the integrand should approach zero as $|S| \to \infty$. Thus, in the region $r \leqslant h$, the appropriate contour of integration is a semi-circular arc which engulfs the second and third quadrants of complex S-plane. However, in the region $r \geqslant h$, a contour must be chosen which contains the first and fourth quadrants of complex S-plane. By utilizing the residue theorem, we obtain the stress field

$$\tau^1_{\theta z}(r,\theta) = \frac{P}{h(R\theta_1+\theta_2)}\sum_n \left(\frac{r}{h}\right)^{S_n-1} \frac{\sin\left[S_n(\theta_1-\theta)\right] + \sin\left[S_n(\theta+\theta_2)\right] + (R-1)\sin(S_n\theta_2)\cos(S_n\theta)}{\sin(S_n\theta_1)\sin(S_n\theta_2)\left[1+\lambda\cot^2(S_n\theta_1)\right]}, \quad r \leqslant h$$

$$\tau^1_{\theta z}(r,\theta) = \frac{P}{h(R\theta_1+\theta_2)}\sum_n \left(\frac{h}{r}\right)^{S_n+1} \frac{\sin\left[S_n(\theta_1-\theta)\right] + \sin\left[S_n(\theta+\theta_2)\right] + (R-1)\sin(S_n\theta_2)\cos(S_n\theta)}{\sin(S_n\theta_1)\sin(S_n\theta_2)\left[1+\lambda\cot^2(S_n\theta_1)\right]}, \quad r \geqslant h \tag{23}$$

in which

$$\lambda = \frac{R(\theta_1 + R\theta_2)}{R\theta_1 + \theta_2} \tag{24}$$

Similar relations may be derived for the stress component τ_{rz} and also the displacement field, which are not given here for the sake of brevity. Moreover, corresponding relations may be written for the other wedge designated by sub/superscript 2. The first relation of (23) shows a strength of geometric singularity of

$$\lambda_S = 1 - S_1 \tag{25}$$

at the wedge apex. As mentioned before, S_1 is the least of the poles S_n.

In the special case, where the apex angles of the two wedges are equal ($\theta_1 = \theta_2 = \alpha$), we obtain from the equation of poles (21) two series of poles which are: $S_n = (n\pi/\alpha)$ and $S_n = [(2n+1)\pi/2\alpha]$ ($n = 0, 1, 2, \ldots$). Applying the first set, (23) give nothing other than zero and with the second set, we obtain

$$\tau^1_{\theta z}(r,\theta) = \frac{P}{h\alpha}\sum_{n=0}^{\infty}(-1)^n\left(\frac{r}{h}\right)^{\frac{(2n+1)\pi}{2\alpha}-1}\cos\left(\frac{(2n+1)\pi\theta}{2\alpha}\right), \quad r \leqslant h$$

$$\tau^1_{\theta z}(r,\theta) = \frac{P}{h\alpha}\sum_{n=0}^{\infty}(-1)^n\left(\frac{h}{r}\right)^{\frac{(2n+1)\pi}{2\alpha}+1}\cos\left(\frac{(2n+1)\pi\theta}{2\alpha}\right), \quad r \geqslant h \tag{26}$$

It may be observed that eqns (26) are independent of material property and thus, for equal shear moduli ($\mu_1 = \mu_2$), the same results are obtained, too. Therefore, these results should be obtained by analyzing an isotropic wedge with apex angle 2α. This analysis has already been carried out by Kargarnovin et al. (1997) and we may obtain the same results as (26) by letting $a \to \infty$ and $h_1 = h_2$ and converting $\alpha \to 2\alpha$ and $\theta \to \theta - \alpha$ in the traction–traction case. From (26), it is seen that the stress field is bounded in a bi-material wedge with $0 < \alpha < (\pi/2)$, whereas in a wedge with $(\pi/2) < \alpha < \pi$, we have a strength of singularity of

$$\lambda_S = 1 - \frac{\pi}{2\alpha} \tag{27}$$

at the wedge apex. Letting $\alpha = \pi$, we encounter the antiplane problem of an edge crack in a composite wedge and we obtain a singularity of the order $1/2$, which is the case for an edge crack.

4. Analysis of a bi-material wedge with an interfacial crack

Existence of an interfacial crack, brings the second integral in (19) into account. To obtain the stress field, contour integration should be carried out. Both the integrands in (19) are meromorphic functions in S and four distinct regions of $0 \leqslant r \leqslant a$, $a \leqslant r \leqslant b$, $b \leqslant r \leqslant h$ and $r \geqslant h$ should be recognized. Since we need only the region $a \leqslant r \leqslant b$ for applying the boundary condition (7), we may carry out the contour integration solely in this zone. For this reason, for the first integral in (19), we complete the contour of integration by a semi-circular arc to include the negative part of the real axis, Re $(S) < 0$. For the second integral in (19), after changing the order of integration we must break the limits of the integral in $[a, b]$ into regions $[a, r]$ and $[r, b]$. Then, for the integral in $[a, r]$ we complete the contour of integration by a semi-circular arc to include the positive part of the real axis, Re $(S) > 0$, however, for the integral in $[r, b]$ a semi-circular arc containing the negative part of the real axis, Re$(S) < 0$, must be considered. Since the integrands in (19) vanish as $|S| \to \infty$, by utilizing the residue theorem, we obtain

$$r\tau^1_{\theta z}(r,\theta) = \frac{1}{R\theta_1 + \theta_2}\left[\sum_n P\left(\frac{r}{h}\right)^{S_n} \frac{\sin\left[S_n(\theta_1-\theta)\right] + \sin\left[S_n(\theta+\theta_2)\right] + (R-1)\sin(S_n\theta_2)\cos(S_n\theta)}{\sin(S_n\theta_1)\sin(S_n\theta_2)\left[1+\lambda\cot^2(S_n\theta_1)\right]}\right.$$
$$\left. - \sum_n \int_a^r \frac{\mu_2 f(v)\left(\frac{v}{r}\right)^{S_n}\sin\left[S_n(\theta_1-\theta)\right]}{\sin(S_n\theta_1)\left[1+\lambda\cot^2(S_n\theta_1)\right]}\,\mathrm{d}v + \sum_n \int_r^b \frac{\mu_2 f(v)\left(\frac{r}{v}\right)^{S_n}\sin\left[S_n(\theta_1-\theta)\right]}{\sin(S_n\theta_1)\left[1+\lambda\cot^2(S_n\theta_1)\right]}\,\mathrm{d}v\right],\quad a\leqslant r\leqslant b \tag{28}$$

in which λ is defined by the same relation (24).

Now, we apply the boundary condition (7) on (28) to obtain

$$\sum_n \frac{\int_a^r \left(\frac{v}{r}\right)^{S_n} f(v)\,\mathrm{d}v}{1+\lambda\cot^2(S_n\theta_1)} - \sum_n \frac{\int_r^b \left(\frac{r}{v}\right)^{S_n} f(v)\,\mathrm{d}v}{1+\lambda\cot^2(S_n\theta_1)} = \frac{P}{\mu_2}\sum_n \frac{\left(\frac{r}{h}\right)^{S_n}\left[\sin(S_n\theta_1) + R\sin(S_n\theta_2)\right]}{\sin(S_n\theta_1)\sin(S_n\theta_2)\left[1+\lambda\cot^2(S_n\theta_1)\right]},\quad a\leqslant r\leqslant b \tag{29}$$

This is the basic relation for the derivation of the singular integral equation.

5. Derivation of the singular integral equation in the case $\theta_1 = \theta_2 = \alpha$

In this case, the equation of the poles (21) reduces to

$$\sin(S_n\alpha)\cos(S_n\alpha) = 0 \tag{30}$$

which gives two sets of poles: $S_n = (n\pi/\alpha)$ and $S_n = (n\pi/\alpha) + (\pi/2\alpha)$ $(n = 0, 1, 2, 3, \ldots)$. Application of the first set, $S_n = (n\pi/\alpha)$, in (29) gives nothing other than zero. Thus, applying $S_n = (n\pi/\alpha) + (\pi/2\alpha)$ in (29) and changing the order of summation and integration, yields

$$\int_a^r \left(\frac{v}{r}\right)^{\frac{\pi}{2\alpha}} \sum_{n=0}^{\infty} \left(\frac{v}{r}\right)^{\frac{n\pi}{\alpha}} f(v)\,\mathrm{d}v - \int_r^b \left(\frac{r}{v}\right)^{\frac{\pi}{2\alpha}} \sum_{n=0}^{\infty} \left(\frac{r}{v}\right)^{\frac{n\pi}{\alpha}} f(v)\,\mathrm{d}v = \frac{P(1+R)}{\mu_2}\left(\frac{r}{h}\right)^{\frac{\pi}{2\alpha}} \sum_{n=0}^{\infty}(-1)^n\left(\frac{r}{h}\right)^{\frac{n\pi}{\alpha}}, \quad a \leqslant r \leqslant b \tag{31}$$

In the first integral of (31), we see that $v < r$ and in the second integral $v > r$. Also, we have supposed that $h \geqslant b$. Therefore, we observe that v/r, r/v and r/h are less than unity and we may utilize the following series expansion formulas:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}; \quad |x| < 1 \tag{32}$$

to obtain

$$\int_a^r \left(\frac{v}{r}\right)^{\gamma/2} \frac{r^{\gamma}}{v^{\gamma} - r^{\gamma}} f(v)\,\mathrm{d}v + \int_r^b \left(\frac{r}{v}\right)^{\gamma/2} \frac{v^{\gamma}}{v^{\gamma} - r^{\gamma}} f(v)\,\mathrm{d}v = -\frac{P(1+R)}{\mu_2} \frac{\left(\frac{r}{h}\right)^{\gamma/2}}{1+\left(\frac{r}{h}\right)^{\gamma}}, \quad a \leqslant r \leqslant b \tag{33}$$

in which

$$\gamma = \frac{\pi}{\alpha} \tag{34}$$

It is seen that the integrands of the first and second integrals in (33) are the same and thus, we can join both integrals to arrive at a single integral in the $[a, b]$ region:

$$\int_a^b \frac{r^{\gamma/2} v^{\gamma/2}}{v^{\gamma} - r^{\gamma}} f(v)\,\mathrm{d}v = -\frac{P(1+R)}{\mu_2} \frac{\left(\frac{r}{h}\right)^{\gamma/2}}{1+\left(\frac{r}{h}\right)^{\gamma}}, \; a \leqslant r \leqslant b \tag{35}$$

If we now make the following change in variables:

$$t = v^{\gamma}, \quad x = r^{\gamma}, \quad c = a^{\gamma}, \quad d = b^{\gamma}, \quad e = h^{\gamma} \tag{36}$$

and define

$$\phi(t) = t^{(1/\gamma)-(1/2)} f\left(t^{1/\gamma}\right) \tag{37}$$

eqn (35) becomes

$$\int_c^d \frac{\phi(t)\,\mathrm{d}t}{t-x} = -\gamma \frac{\mu_1 + \mu_2}{\mu_1 \mu_2} P \frac{\sqrt{e}}{e+x}; \quad c \leqslant x \leqslant d \tag{38}$$

Eqn (38) has to be solved as it is subjected to the condition (16), which in the light of (36) and (37) changes form to

$$\int_c^d \frac{\phi(t)}{t^{1/2}}\,\mathrm{d}t = 0 \tag{39}$$

Eqns (38) and (39) are very similar in form with those obtained by Erdogan and Gupta (1975). Here, we have solved it analytically. The general solution of eqn (38) which has integrable singularities at the ends

c and d may be written as (Muskhelishvili, 1977)

$$\phi(x) = \left[(x-c)\,(d-x)\right]^{-1/2}\left[B - \frac{1}{\pi^2}\int_c^d \left[(t-c)\,(d-t)\right]^{1/2}\frac{P(t)}{t-x}\,\mathrm{d}t\right]; \quad c \leqslant x \leqslant d \tag{40}$$

in which

$$P(t) = -\gamma\frac{\mu_1+\mu_2}{\mu_1\mu_2}P\frac{\sqrt{e}}{e+t} \tag{41}$$

To obtain $\phi(x)$, we must compute the integral

$$I = \int_c^d \frac{\sqrt{(t-c)\,(d-t)}\,\mathrm{d}t}{(t-x)\,(t+e)} \tag{42}$$

The term x in (42) should be regarded as a constant with respect to the integration variable, t. To compute this integral, we may use the following change in variable:

$$\sqrt{(t-c)\,(d-t)} = (t-c)u \tag{43}$$

Applying this change of variable on (42), we obtain

$$I = \frac{2(d-c)^2}{(c-x)\,(c+e)}\int_0^\infty \frac{u^2\,\mathrm{d}u}{(u^2+1)\,(u^2-m^2)\,(u^2+l^2)} \tag{44}$$

where

$$m^2 = \frac{d-x}{x-c}$$

$$l^2 = \frac{d+e}{c+e} \tag{45}$$

Now, it is possible to compute the integral in (44) by the method of partial fraction expansion. Substituting the results into (40) and facilitating yields

$$\phi(x) = \left[B - \frac{P\sqrt{e}}{\mu_e\alpha} + \frac{P\sqrt{e(c+e)\,(d+e)}}{\mu_e\alpha(x+e)}\right]\left[(x-c)\,(d-x)\right]^{-1/2}, \quad c \leqslant x \leqslant d \tag{46}$$

in which $\mu_e = [\mu_1\mu_2/(\mu_1+\mu_2)]$.

The constant B should be obtained by applying the condition (39) on (46), which gives

$$B = \frac{P\sqrt{e}}{\mu_e\alpha}\left[1 - \sqrt{(c+e)\,(d+e)}\frac{I_1}{I_2}\right] \tag{47}$$

where

$$I_1 = \int_c^d \frac{\mathrm{d}t}{(t+e)\sqrt{t(t-c)\,(d-t)}}$$

$$I_2 = \int_c^d \frac{dt}{\sqrt{t(t-c)(d-t)}} \tag{48}$$

To carry out the integrations in (48), we make the successive changes in variables:

$$t = (d-c)u^2 + c$$

$$u = \cos\theta \tag{49}$$

and then after substituting the results together with (47) into (46), we arrive at

$$\phi(x) = \frac{P}{\mu_e \alpha}\sqrt{e(c+e)(d+e)}\left[\frac{1}{x+e} - \frac{1}{d+e}\frac{\Pi\left(k, n, \frac{\pi}{2}\right)}{K\left(k, \frac{\pi}{2}\right)}\right]\left[(x-c)(d-x)\right]^{-1/2}; \quad c \leqslant x \leqslant d \tag{50}$$

where

$$k^2 = \frac{d-c}{d}$$

$$n = -\frac{d-c}{d+e} \tag{51}$$

and $K(k, \pi/2)$ and $\Pi(k, n, \pi/2)$ are the complete elliptic integrals of the first and third kinds, respectively (Spiegel, 1968).

Now, with the aid of (36) and (37), we may obtain $f(r)$ as follows:

$$f(r) = \frac{P}{\mu_e \alpha}\sqrt{h^\gamma(a^\gamma + h^\gamma)(b^\gamma + h^\gamma)}\left[\frac{1}{r^\gamma + h^\gamma} - \frac{1}{b^\gamma + h^\gamma}\frac{\Pi\left(k, n, \frac{\pi}{2}\right)}{K\left(k, \frac{\pi}{2}\right)}\right] r^{(\gamma/2)-1}\left[(r^\gamma - a^\gamma)(b^\gamma - r^\gamma)\right]^{-1/2}; \quad a \leqslant r \leqslant b \tag{52}$$

where

$$k^2 = \frac{b^\gamma - a^\gamma}{b^\gamma}, \quad n = -\frac{b^\gamma - a^\gamma}{b^\gamma + h^\gamma} \tag{53}$$

If the location of application of the concentrated traction is far enough from the crack tips ($h \gg b$), from the second of (53) we see that $n \rightarrow 0$. Since

$$\Pi\left(k, 0, \frac{\pi}{n}\right) = K\left(k, \frac{\pi}{2}\right) \tag{54}$$

we will have from (52) that $f(r) \rightarrow 0$. On the other hand, if the crack tip, $r = a$, coincides with the wedge apex ($a = 0$), then a singularity of the order unity is observed at the apex.

The stress intensity factors may be defined as was done by Erdogan (1966) and Erdogan and Gupta (1975):

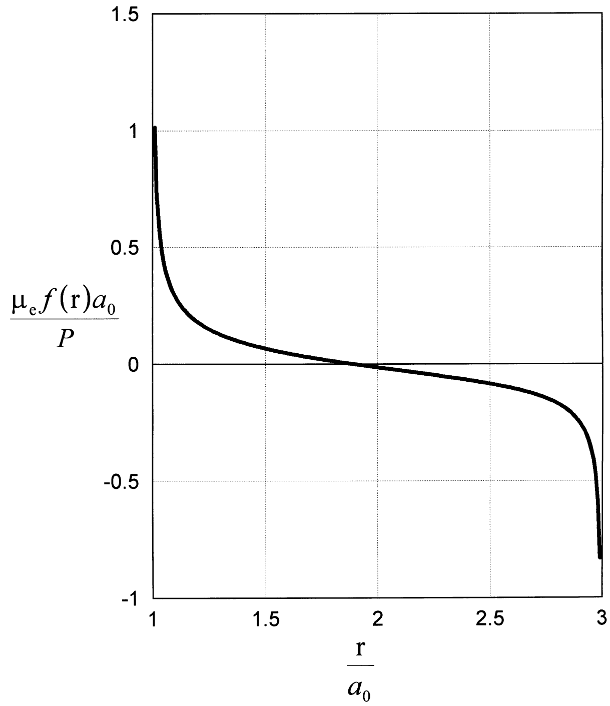

Fig. 2. Variations of $f(r)$ as a function of r/a_0 $(a \leqslant r \leqslant b)$.

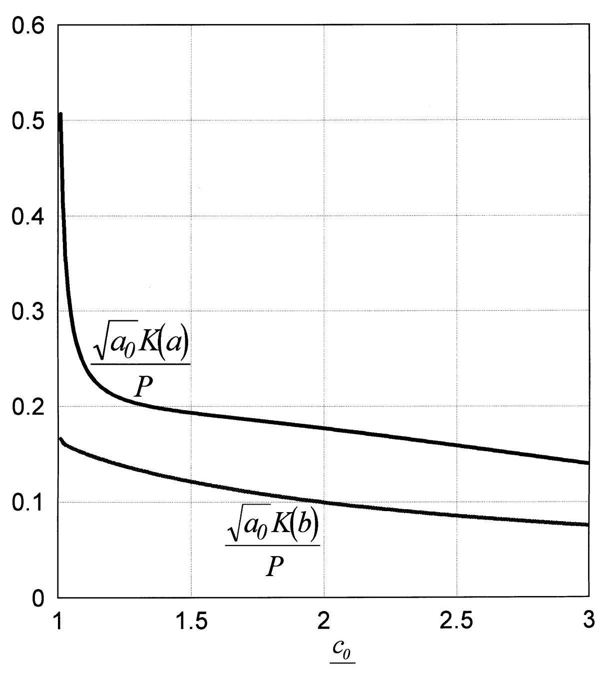

Fig. 3. Variations of the stress intensity factors as functions of relative crack distance.

$$K(a) = \lim_{r \to a} \left[2(a - r)\right]^{1/2} \tau_{\theta z}^{1}(r, 0) = \lim_{r \to a} \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} \left[2(r - a)\right]^{1/2} f(r)$$

$$K(b) = \lim_{r \to b} \left[2(r - b)\right]^{1/2} \tau_{\theta z}^{1}(r, 0) = -\lim_{r \to b} \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} \left[2(b - r)\right]^{1/2} f(r) \tag{55}$$

In order to take the location and the length of the crack into account, we may write the distance of the centre of the crack with respect to the apex and the crack length in terms of the crack tips coordinates

$$c_0 = \frac{a + b}{2}$$

$$a_0 = \frac{b - a}{2} \tag{56}$$

Substituting (56) into (52), (53) and (55), we may show variations of the density function $f(r)$ and the stress intensity factors in appropriate graphs. Fig. 2 shows the variation of $\mu_e f(r) a_0 / P$ as a function of r/a_0 in constant c_0/a_0 and h/a_0. In this particular example $\alpha = \pi/2$, $c_0/a_0 = 2$ and $h/a_0 = 4$. Fig. 3 shows the variation of the stress intensity factors as functions of relative crack distance, c_0/a_0, for $\alpha = \pi/2$ and $h/a_0 = 4$. It is observed that $K(a) \to 0$, $K(b) \to 0$ as $c_0/a_0 \to \infty$. On the other hand, $K(a) \to \infty$ as $c_0/a_0 \to 1$ (or $a \to 0$), whereas $K(b)$ is finite for $c_0/a_0 = 1$ (or $a = 0$).

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